STiCM

Select / Special Topics in Classical Mechanics

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STiCM Lecture 13: Unit 4

Symmetry and Conservation Laws

Dynamical Symmetry in the Kepler Problem

Unit 1 : Equations of Motion (Newton / Lagrange / Hamilton)

Unit 2: Oscillators (Free/damped/driven), Resonances, Waves

Unit 3: Polar Coordinate Systems

Symmetry $\longleftrightarrow$ Conservation laws (Noether)

Unit 4 : Dynamical Symmetry in the Kepler Problem

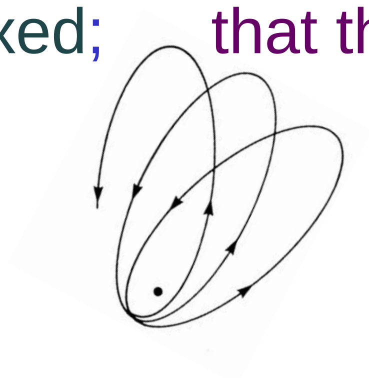
Unit 4: Learning goals: Recapitulate:

Conservation of energy that is well-known in the Kepler-Bohr problem stems from the symmetry with regard to temporal translations (displacements on the time axis).

GETTING CONSERVATION LAWS FROM THE EQUATION OF MOTION

Conservation of angular momentum, likewise, stems from the central field symmetry in the Kepler problem.

Neither of these accounts for the fact that the Kepler ellipse remains fixed; that the ellipse does *not* undergo a ‘rosette’ motion.

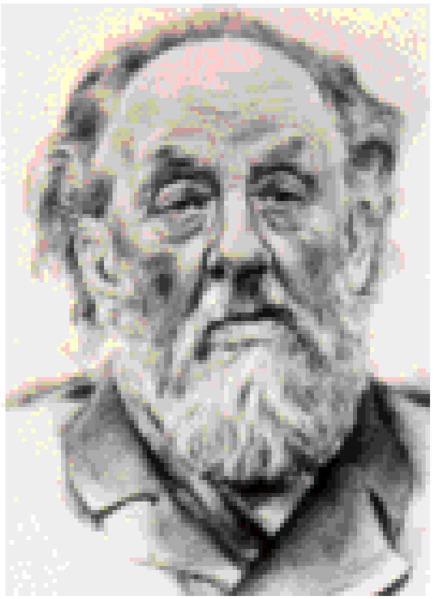


This unit will discover the ‘dynamical’ symmetry of the Kepler problem *and* its relation with the constancy (conservation) of the LRL vector, which keeps the orbit ‘fixed’.

Motivation: Connection between symmetry & conservation laws has important consequences on issues at the very frontiers of physics and technology.

Mechanics of Flights into Space

Mechanics of Flights into Space



Konstantin Tsiolkovsky
(1857-1935)

Robert H. Goddard
(1882-1945)

Hermann
Oberth
(1894-1989)

Indian Space Research Organisation (ISRO)

[1] Indian National Satellites (INSAT)

- for communication services

[2] Indian Remote Sensing (IRS) Satellites

- for management of natural resources

[3] Polar Satellite Launch Vehicle (PSLV)

- for launching IRS type of satellites

[4] Geostationary Satellite Launch Vehicle (GSLV)

- for launching INSAT type of satellites.

Gravity plays the most important role in designing satellite trajectories,

of course,

and hence we study the Kepler TWO-BODY problem

We must then adapt the formalism to understand the models, methods and applications of satellite orbits, etc.

Other than 'energy' and 'angular momentum', what *else* is conserved, *and* what is the associated symmetry?

For given physical laws of nature, what quantities are conserved?

Rather, if you can observe what physical quantities are conserved, can you discover the physical laws of nature?

How did Kepler deduce that planetary orbits are ellipses around the sun ?

Johannes Kepler
1571- 1630

How would you solve this problem?

Kepler had no knowledge of :

(a) differential equations

(b) inverse square force (gravity).

How did Kepler deduce that planetary orbits are ellipses around the sun ?

Kepler got his elliptic orbits not by solving differential equations for gravitational force, but by doing clever curve fitting of Tycho Brahe's experimental data.

Tycho Brahe (1546-1601):

Danish astronomer appointed as the imperial astronomer under Rudolf II in Prague, which then came under the Roman empire.

Brahe had his own “Tychonian” model of planetary motion:

Brahe had planets revolve around the sun (like the Copernican system), and the sun and the moon going around the earth (like the system of Ptolemy).

What was Brahe’s nose was made of ?

Tycho Brahe (1546-1601):

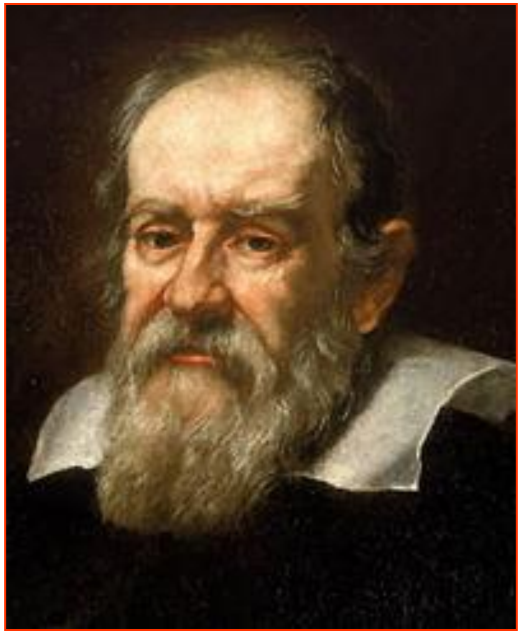
He discovered the supernova in 1572, in "Cassiopeia'.

Kepler and Brahe never could collaborate successfully.

They quarreled, and Tycho did not provide Kepler any access to the high precision observational data he (Brahe) had complied.
It was only on his deathbed, saying
"……let me not seem to have lived in vain……."
that Brahe handed over his observational data to Kepler [Ref.:Sagan].

Johannes Kepler
1571- 1630

Galileo Galilei
1564 - 1642

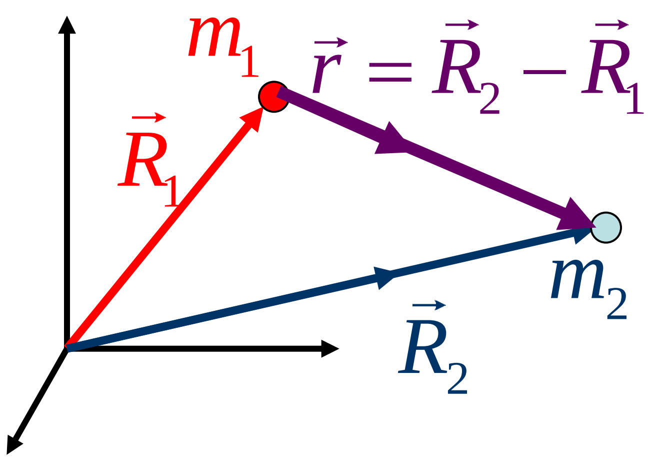


Isaac Newton
(1642-1727)

Causality, Determinism, Equation of Motion

'Dynamics' came well AFTER KEPLER!

Two-body problem



$$\hat{u} = \frac{\vec{r}}{|\vec{r}|}$$

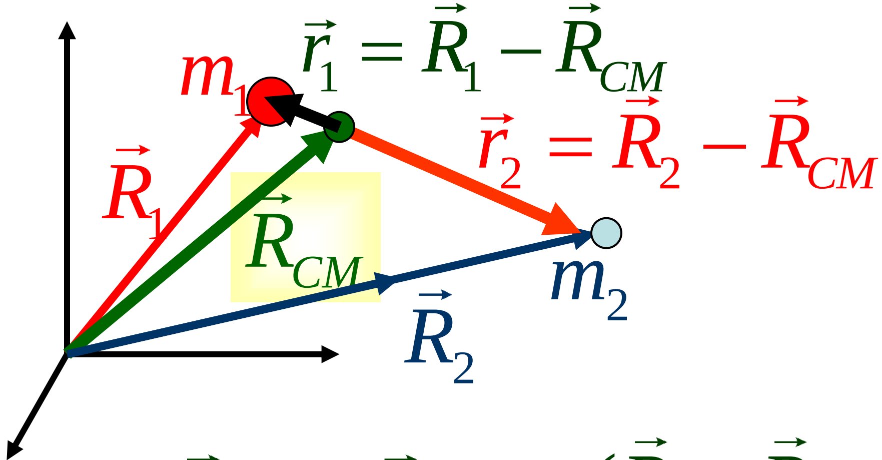
$$\vec{F}_{by\_1\_on\_2}$$

$$m_2 \ddot{\vec{R}}_2 = -G \frac{m_1 m_2}{\left|\vec{R}_2 - \vec{R}_1\right|^2} \hat{u}$$

$$\vec{F}_{by\_2\_on\_1}$$

$$m_1 \ddot{\vec{R}}_1 = G \frac{m_1 m_2}{\left|\vec{R}_2 - \vec{R}_1\right|^2} \hat{u}$$

Two-body problem: Centre of Mass



$$\vec{R}_{CM} = \frac{m_1\vec{R}_1 + m_2\vec{R}_2}{m_1 + m_2}$$

$$m_1 \rangle\rangle\rangle m_2$$

$$m_1\vec{r}_1 + m_2\vec{r}_2 = m_1(\vec{R}_1 - \vec{R}_{CM}) + m_2(\vec{R}_2 - \vec{R}_{CM})$$
$$= m_1\vec{R}_1 + m_2\vec{R}_2 - \left(m_1 + m_2\right)\vec{R}_{CM}$$
$$= \vec{0}.$$

$$\vec{r} = \vec{R}_2 - \vec{R}_1 = \vec{r}_2 - \vec{r}_1$$

$$\vec{F}_{by\_1\_on\_2}$$

$$m_2\ddot{\vec{R}}_2 = -G\frac{m_1 m_2}{\left|\vec{R}_2 - \vec{R}_1\right|^2}\hat{u}$$

$$\hat{u} = \frac{\vec{r}}{\left|\vec{r}\right|}$$

$$\vec{F}_{by\_2\_on\_1}$$

$$m_1\ddot{\vec{R}}_1 = G\frac{m_1 m_2}{\left|\vec{R}_2 - \vec{R}_1\right|^2}\hat{u}$$

$$\ddot{\vec{r}} = \ddot{\vec{R}}_2 - \ddot{\vec{R}}_1 = -G\left(m_1 + m_2\right)\frac{\vec{r}}{\left|\vec{r}\right|^3}$$

$$\vec{r} = \vec{R}_2 - \vec{R}_1 = \vec{r}_2 - \vec{r}_1$$

$$= -\kappa\frac{\vec{r}}{\left|\vec{r}\right|^3}\text{...where } \kappa = G\left(m_1 + m_2\right) \approx Gm_1\text{....for } m_1 \rangle\rangle m_2$$

$$\ddot{\vec{r}} + \kappa\frac{\vec{r}}{\left|\vec{r}\right|^3} = \vec{0}.$$

$$\left[\kappa\right] = L^3 T^{-2}$$

This equation of motion describes the 'relative motion' of the smaller mass relative to the larger mass, *assuming that the difference in the masses is huge*.

$$\ddot{\vec{r}} + \kappa \frac{\vec{r}}{|\vec{r}|^3} = \vec{0}.$$

We now take the dot product of the velocity with the 'Eq. of motion':

$$\dot{\vec{r}} \bullet \ddot{\vec{r}} + \frac{\kappa}{|\vec{r}|^3} \dot{\vec{r}} \bullet \vec{r} = 0$$

$$\dot{\vec{r}} = \frac{d}{dt}\vec{r} = \frac{d}{dt}(r\hat{u}) = \dot{r}\hat{u} + r\dot{\hat{u}}$$

$$\dot{\vec{r}} \bullet \ddot{\vec{r}} = \vec{v} \bullet \dot{\vec{v}} = v\dot{v}$$

$$\dot{\vec{r}} \bullet \vec{r} = (\dot{r}\hat{u} + r\dot{\hat{u}}) \bullet (r\hat{u}) = r\dot{r}$$

$$v\dot{v} + \frac{\kappa}{r^3} r\dot{r} = 0$$

$$v \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = -\frac{\kappa \lim_{\delta t \to 0} \frac{\delta r}{\delta t}}{r^2}$$

$$\ddot{\vec{r}} + \kappa \frac{\vec{r}}{|\vec{r}|^3} = \vec{0}.$$

We took the dot product of the 'Eq. of motion' with velocity:

$$\mathrm{v} \lim_{\delta t \to 0} \frac{\delta \mathrm{v}}{\delta t} = -\frac{\kappa \lim_{\delta t \to 0} \frac{\delta r}{\delta t}}{r^2}$$

Integration w.r.t. time

Differentiation w.r.t. time

$$\frac{\mathrm{v}^2}{2} = \frac{\kappa}{r} + E$$

$$\frac{\mathrm{v}^2}{2} - \frac{\kappa}{r} = E$$

DIMENSIONS

$$[E] = L^2 T^{-2}$$

$$[\kappa] = L^3 T^{-2}$$

Total 'SPECIFIC' (*i.e. per unit mass*) MECHANICAL ENERGY: Constant of integral / Constant of Motion. INVARIANCE, SYMMETRY.

$$\ddot{\vec{r}} + \kappa \frac{\vec{r}}{|\vec{r}|^3} = \vec{0}.$$

Equation of Motion

We took the dot product of the 'Eq. of motion' with velocity:

$$\dot{\vec{r}} \bullet \ddot{\vec{r}} + \frac{\kappa}{|\vec{r}|^3} \dot{\vec{r}} \bullet \vec{r} = 0$$

…….. and discovered a CONSERVED QUANTITY!

$$\frac{\mathrm{v}^2}{2} - \frac{\kappa}{r} = E$$

E: constant

symmetry with respect to translations in time.

(E,t): canonically conjugate pair of variables

We now take the cross product of the position vector with the 'Eq. of motion':

$$\vec{r} \times \ddot{\vec{r}} + \vec{r} \times \kappa \frac{\vec{r}}{|\vec{r}|^3} = \vec{0}.$$

$$Therefore: \ \vec{r} \times \ddot{\vec{r}} = \vec{0}$$

Force: RADIAL

central force symmetry

$$Now, \frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}} = \vec{r} \times \ddot{\vec{r}} = \vec{0}$$

Therefore $\vec{H} = \vec{r} \times \dot{\vec{r}} = \vec{r} \times \vec{v}$ is also a constant of motion.

SPECIFIC ANGULAR MOMENTUM

INVARIANCE, SYMMETRY

Homogeneity of time ⟷ Conservation of energy

Homogeneity of space
Translational Symmetry ⟷ Conservation of linear momentum

Isotropy of space
Rotational Symmetry ⟷ Conservation of angular momentum

Emmy Noether
1882 to 1935

SYMMETRY ⟷ CONSERVATION LAWS

Her entry to the Senate of the University of Gottingen, Germany, was resisted.

David Hilbert argued in favor of admitting her to the University Senate.

Recapitulate…
… From Unit 1

$$\frac{\partial L}{\partial t} = 0$$

Time is homogeneous: Lagrangian of a closed system does not depend explicitly on time.

$$\left[\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right]$$ is a CONSTANT.

Hamiltonian

"ENERGY"

Conservation of Energy is thus connected with the symmetry principle regarding invariance with respect to temporal translations.

Hamiltonian / Hamilton's Principal Function

Space is homogenous and isotropic

Recapitulate...
... From Unit 1

$$\text{the condition for homogeneity of space : } \delta L(x, y, z) = 0$$

$$i.e., \quad \delta L = \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial z}\delta z = 0$$

$$\text{which implies } \frac{\partial L}{\partial q} = 0 \text{ where } q = x, y, z$$

$$\text{since } \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0, \text{ this means } i.e. \ \frac{\partial L}{\partial \dot{q}} = p \text{ is conserved.}$$

i.e., is independent of time, is a constant of motion

Law of conservation of momentum,
arises from the homogeneity of space.

Symmetry ⟷ Conservation Laws

Recapitulate…
… From Unit 1

$$0 = \sum_{k=1}^{N} \vec{F}_k^{(i)} \bullet \overrightarrow{\delta S}$$

Displacement is arbitrary, hence, $\sum_{k=1}^{N} \vec{F}_k^{(i)} = \vec{0};$

$$i.e.\ \vec{0} = \frac{d}{dt} \sum_{k=1}^{N} \vec{p}_k^{(i)} = \frac{d}{dt}\vec{P}$$

Thus $\vec{P}$ is conserved in the absence of external forces.

The conservation of momentum was secured in 'Unit 1' on the basis of 'translational invariance in homegenous space', and *not* on the basis of Newton's III law.

$$\frac{d\overrightarrow{P}}{dt} = \vec{0}$$

$$\frac{d\overrightarrow{p_2}}{dt} = -\frac{d\overrightarrow{p_1}}{dt}$$

$$\overrightarrow{F}_{12} = -\overrightarrow{F}_{21}$$

Given the symmetry related to translation in homogenous space, could you have discovered Newton’s 3rd law?

Are the conservation principles consequences of the laws of nature? Or, are the laws of nature the consequences of the symmetry principles that govern them?

Until Einstein's special theory of relativity,
it was believed that
conservation principles are the result of the laws of nature.

Since Einstein's work, however, physicists began to analyze the conservation principles as consequences of certain underlying symmetry considerations,
enabling the laws of nature to be revealed from this analysis.

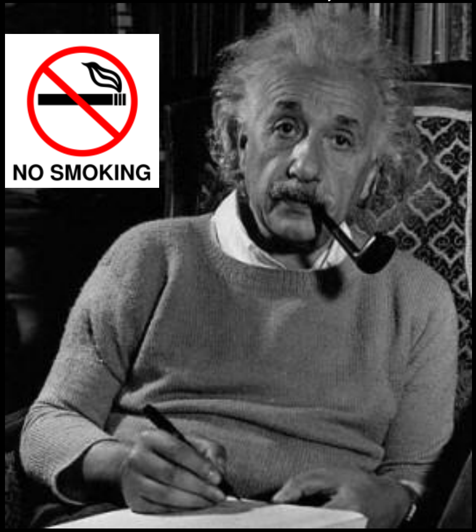
Instead of introducing Newton's III law as a *fundamental principle,*

we *deduced* it (in Unit 1) from symmetry / invariance.

This approach places SYMMETRY *ahead of* LAWS OF NATURE.

It is this approach that is of greatest value to contemporary physics. This approach has its origins in the works of

Albert Einstein, Emmily Noether and Eugene Wigner.



(1879 – 1955)

(1882 – 1935)

(1902 – 1995)

$$\ddot{\vec{r}} + \kappa \frac{\vec{r}}{|\vec{r}|^3} = \vec{0}.$$

Equation of Motion for Kepler's two-body problem

$$\vec{H} = \vec{r} \times \dot{\vec{r}} = \vec{r} \times \vec{\mathrm{v}},$$

the SPECIFIC ANGULAR MOMENTUM

We shall get yet another constant of motion, a conserved quantity, by taking the cross product of the 'SPECIFIC ANGULAR MOMENTUM $\vec{H}$ with the equation of motion:

$$\vec{H} \times \ddot{\vec{r}} + \vec{H} \times \left(\kappa \frac{\vec{r}}{|\vec{r}|^3} \right) = \vec{0}.$$

$$\vec{H}\times\ddot{\vec{r}}+\vec{H}\times\left(\kappa\frac{\vec{r}}{|\vec{r}|^3}\right)=\vec{0}.$$

$$\vec{H}\times\ddot{\vec{r}}+\frac{\kappa}{|\vec{r}|^3}(\vec{r}\times\vec{v})\times\vec{r}=\vec{0}$$

$$\vec{H}\times\ddot{\vec{r}}+\frac{\kappa}{|\vec{r}|^3}\left\{(\vec{r}\bullet\vec{r})\vec{v}-(\vec{r}\bullet\vec{v})\vec{r}\right\}=\vec{0}$$

$$Use:\vec{r}\bullet\vec{v}=\vec{r}\bullet\frac{d}{dt}\left\{r\hat{e}_\rho\right\}=\vec{r}\bullet\hat{e}_\rho\frac{dr}{dt}=r\dot{r}$$

$$\vec{H}\times\ddot{\vec{r}}+\frac{\kappa}{|\vec{r}|^3}\left(r^2\vec{v}-r\dot{r}\vec{r}\right)=\vec{0}$$

$$\boxed{\vec{H}\times\ddot{\vec{r}}}+\frac{\kappa}{|\vec{r}|^3}\left(r^2\vec{\mathrm{v}}-r\dot{r}\vec{r}\right)=\vec{0}$$

$$\text{Now, }\frac{d}{dt}\left(\vec{H}\times\vec{\mathrm{v}}\right)=\frac{d}{dt}\left(\vec{H}\times\dot{\vec{r}}\right)=\boxed{\vec{H}\times\ddot{\vec{r}}}\text{.. since }\dot{\vec{H}}=\vec{0}$$

$$\frac{d}{dt}\left(\vec{H}\times\vec{\mathrm{v}}\right)+\frac{\kappa}{|\vec{r}|^3}\left(r^2\vec{\mathrm{v}}-r^2\dot{r}\hat{e}_\rho\right)=\vec{0}$$

$$\frac{d}{dt}\left(\vec{H}\times\vec{\mathrm{v}}\right)+\kappa\left(\frac{\vec{\mathrm{v}}}{r}-\frac{\dot{r}}{r^2}\vec{r}\right)=\vec{0}$$

$$\frac{d}{dt}\left(\vec{H}\times\vec{\mathrm{v}}\right)+\kappa\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)=\vec{0}$$

$$\frac{d}{dt}\left[\left(\vec{H}\times\vec{\mathrm{v}}\right)+\kappa\left(\frac{\vec{r}}{r}\right)\right]=\vec{0}$$

$$\frac{d}{dt}\left[\left(\vec{H}\times\vec{\mathrm{v}}\right)+\kappa\frac{\vec{r}}{r}\right]=\vec{0}$$

$$\left[\left(\vec{H}\times\vec{\mathrm{v}}\right)+\kappa\hat{e}_{\rho}\right]=-\vec{A},$$

constant

$$\vec{A}=\left(\vec{\mathrm{v}}\times\vec{H}\right)-\kappa\hat{e}_{\rho},\ \text{constant}$$

LAPLACE – RUNGE – LENZ VECTOR

Observe how a constant of motion has emerged – - yet again -– by playing with the equation of motion!

Physical Dimensions

$$\left[\kappa\right]=L^3T^{-2}$$

$$\left[\vec{\mathrm{v}}\times\vec{H}\right]=LT^{-1}\times L^2T^{-1}=L^3T^{-2}$$

We will take a Break…

……. Any questions ?

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References:

[1] Oliver Montenbruck and Eberhard Gill
'Satellite orbits – Models, Methods, Applications'.
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'Introduction to Space Flight'
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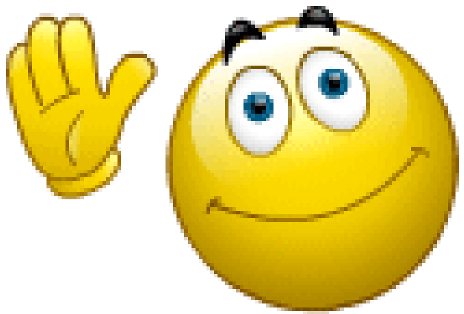

Next Lecture: Dynamical Symmetry of the Kepler Problem

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STiCM Lecture 14: Unit 4

Symmetry and Conservation Laws
Dynamical Symmetry in the Kepler Problem

$$\left(\vec{\mathrm{v}} \times \vec{H}\right) - \kappa \hat{e}_\rho = \vec{A}, \text{ constant}$$

LAPLACE – RUNGE – LENZ VECTOR

Take dot product with $\vec{r}$:

→Equation to the orbit / trajectory

→Without solving the differential equation of motion!

$$\left(\vec{\mathrm{v}}\times\vec{H}\right)-\kappa\hat{e}_{\rho}=\vec{A}$$ where $\vec{H}=\vec{r}\times\vec{\mathrm{v}}$

Take dot product with $\vec{r}$:

$$\left(\vec{\mathrm{v}}\times\vec{H}\right)\bullet\vec{r}-\kappa r=\vec{A}\bullet\vec{r}$$

sign reversal : $\left(\vec{H}\times\vec{\mathrm{v}}\right)\bullet\vec{r}+\kappa r=-\vec{A}\bullet\vec{r}$

Interchange 'dot' and 'cross': $\vec{H}\bullet\left(\vec{\mathrm{v}}\times\vec{r}\right)+\kappa r=-\vec{A}\bullet\vec{r}$

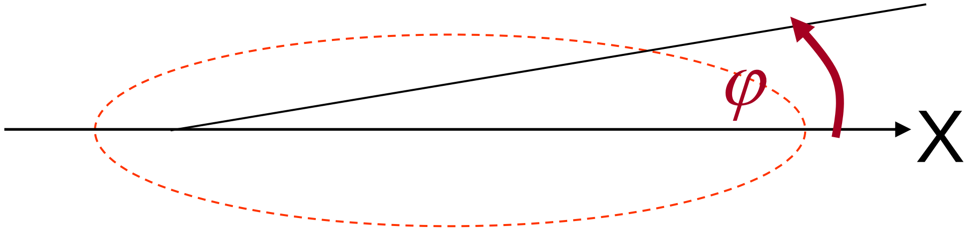
$$-H^{2}+\kappa r=-\vec{A}\bullet\vec{r}=-Ar\cos\varphi$$

$$\varphi=\angle\left(\vec{A},\vec{r}\right)$$

$$-H^2 + \kappa r = -Ar\cos\varphi$$

$$\varphi = \angle\left(\vec{A}, \vec{r}\right)$$

$$\varphi = \angle\left(\vec{A}, \vec{r}\right)$$

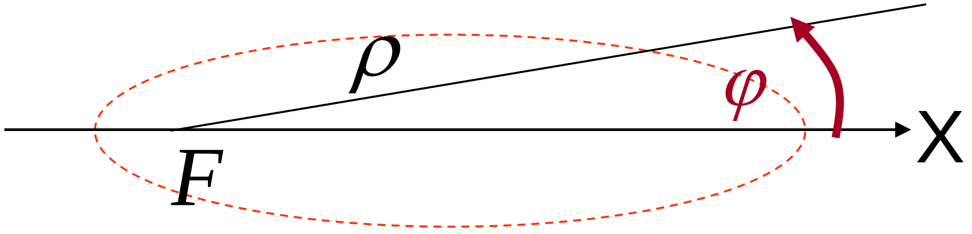


$$r\left(\kappa + A\cos\varphi\right) = H^2$$

$$r = \frac{H^2}{\kappa + A\cos\varphi} = \frac{\left(H^2/\kappa\right)}{1+\left(A/\kappa\right)\cos\varphi} = \frac{p}{1+\varepsilon\cos\varphi}$$

What is $\varepsilon = \left(A/\kappa\right)$?

$$\varphi = \angle\left(\vec{A}, \vec{r}\right)$$



$$\left[\left(\vec{v}\times\vec{H}\right)-\kappa\left(\frac{\vec{r}}{r}\right)\right]=\vec{A}$$

$$\vec{H}=\vec{r}\times\vec{v}\Rightarrow\angle\left(\vec{H},\vec{r}\right)=90^0\ \&\ \angle\left(\vec{H},\vec{v}\right)=90^0$$

$$\vec{H}\times\vec{v}=Hv\hat{u}$$

$$\left(Hv\right)^2+\frac{2\kappa}{r}\left(\vec{H}\times\vec{v}\right)\bullet\vec{r}+\kappa^2=A^2$$

$$\left(\vec{H}\times\vec{v}\right)=\left(\vec{r}\times\vec{v}\right)\times\vec{v}=\left(\vec{v}\bullet\vec{r}\right)\vec{v}-v^2\vec{r}$$

$$\left(\vec{H}\times\vec{v}\right)\bullet\vec{r}=\left(\vec{v}\bullet\vec{r}\right)\vec{v}\bullet\vec{r}-v^2\vec{r}\bullet\vec{r}=\left(vr\cos\xi\right)^2-v^2r^2$$

$$=-v^2r^2\sin^2\xi$$

$$\xi=\angle\left(\vec{r},\vec{v}\right)$$

$$H^2v^2-\frac{2\kappa}{r}v^2r^2\sin^2\xi+\kappa^2=A^2$$

$$H^2 \mathrm{v}^2 - \frac{2\kappa}{r} \mathrm{v}^2 r^2 \sin^2 \xi + \kappa^2 = A^2$$

$$H^2 \left[\mathrm{v}^2 - \frac{2\kappa}{rH^2} \mathrm{v}^2 r^2 \sin^2 \xi \right] + \kappa^2 = A^2$$

$$\vec{H} = \vec{r} \times \vec{\mathrm{v}}$$

$$H^2 \left[\mathrm{v}^2 - \frac{2\kappa}{r} \right] + \kappa^2 = A^2$$

$$H^2 [2E] + \kappa^2 = A^2$$

$$\frac{H^2 [2E]}{\kappa^2} + 1 = \frac{A^2}{\kappa^2}$$

$$\frac{A}{\kappa} = \sqrt{\frac{2EH^2}{\kappa^2} + 1}$$

$$r = \frac{p}{1 + \varepsilon \cos \varphi}$$

with

$$p = \frac{H^2}{\kappa} \ \& \ \varepsilon = \frac{A}{\kappa} = \sqrt{1 + \frac{2EH^2}{\kappa^2}}$$

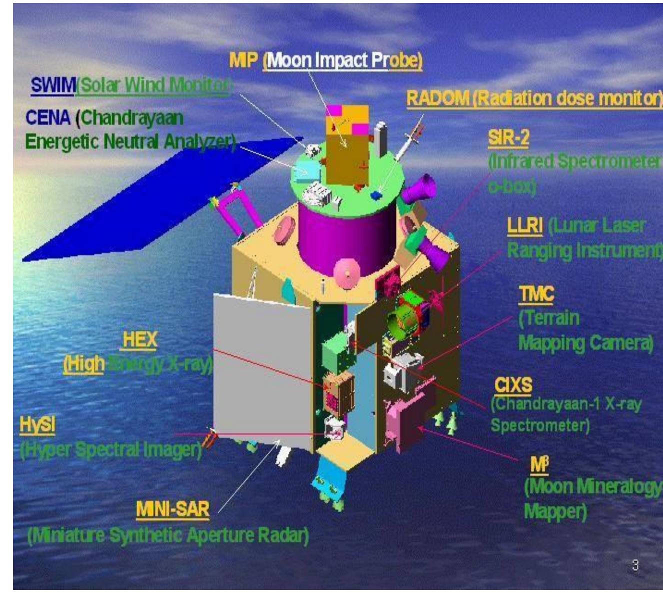
$$r = \frac{p}{1 + \varepsilon \cos\varphi}$$

$\varepsilon \rangle 1$: Hyperbola (open trajectory)

$\varepsilon = 1$: Parabola (open trajectory)

$0 \langle \varepsilon \langle 1$: Ellipse (closed trajectory)

$\varepsilon = 0$: Circle ⟶ degenerate ellipse



For satellite and ballistic missile trajectories, ellipses (inclusive of the circle) are of primary interest.

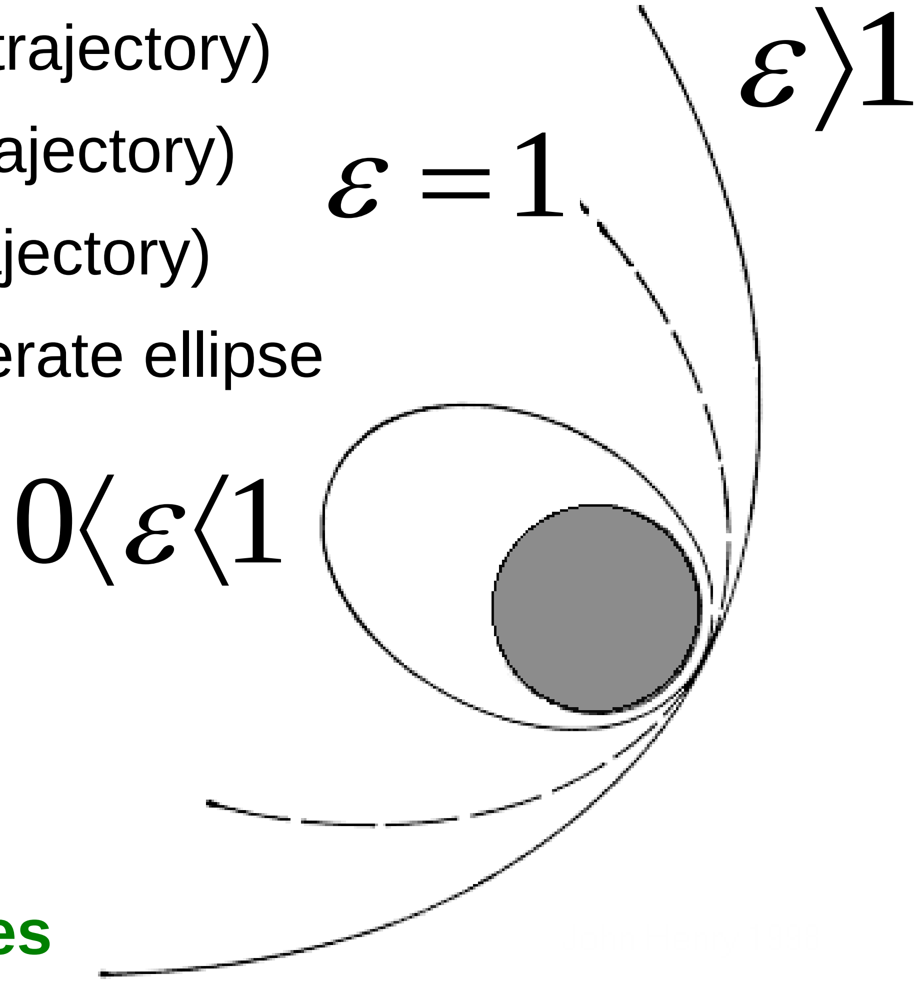
$$r = \frac{p}{1+\varepsilon\cos\varphi}$$

$\varepsilon \rangle 1:$ Hyperbola (open trajectory)

$\varepsilon = 1:$ Parabola (open trajectory)

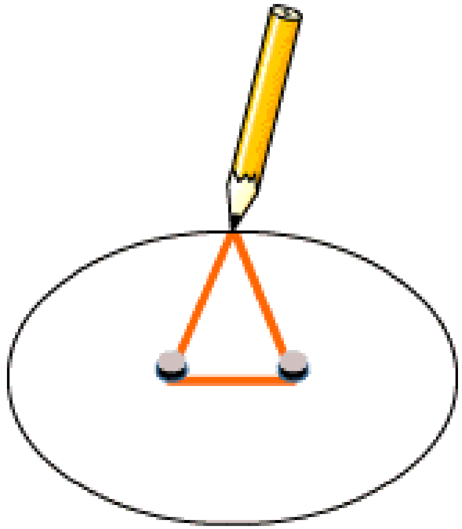
$0 \langle \varepsilon \langle 1:$ Ellipse (closed trajectory)

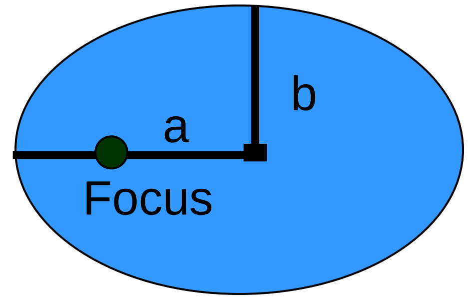
$\varepsilon = 0:$ Circle ⟶ degenerate ellipse

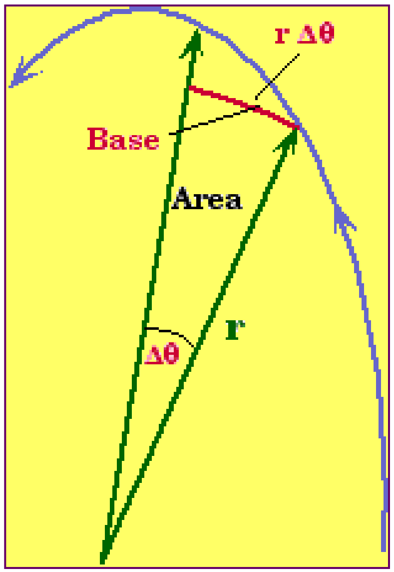


Earth-orbiting satellites are in elliptic motion.

Deep space probes leave earth's gravity on hyperbolic orbits



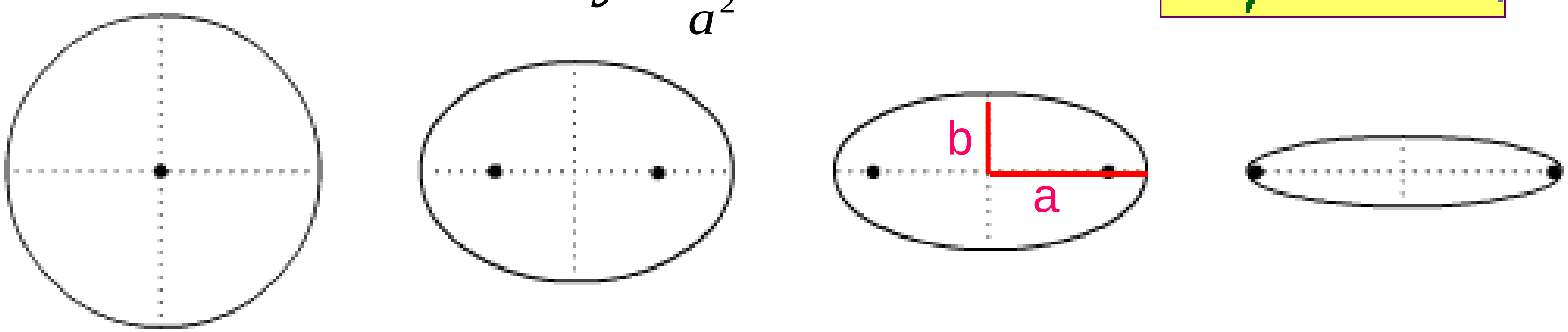




$a =$ semi-major axis

$b =$ semi-minor axis

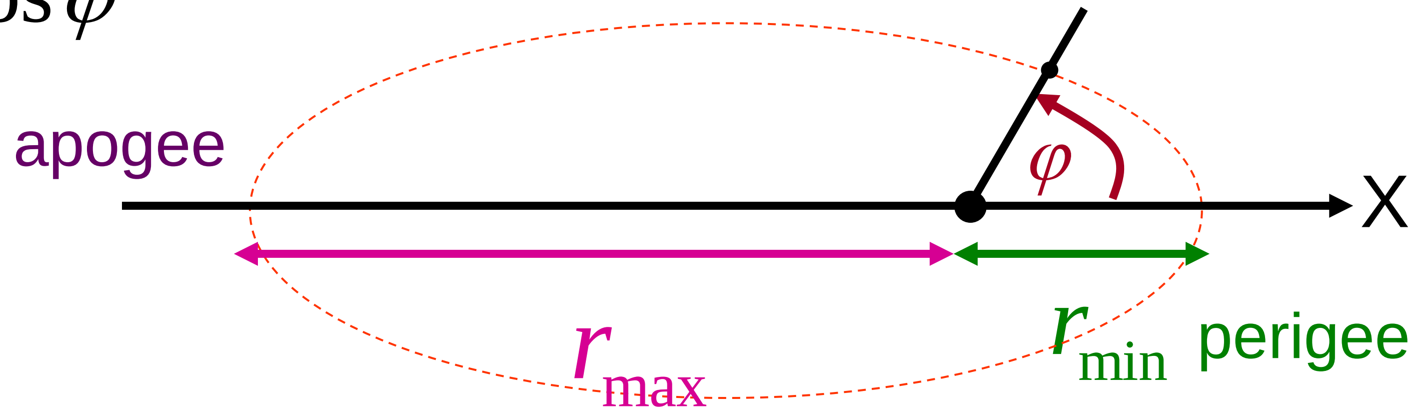
$e : eccentricity \quad \frac{b^2}{a^2} = 1 - e$



e=0 e=0.5 e=0.75 e=0.95

$$r = \frac{p}{1+\varepsilon\cos\varphi};$$

$$0 \le \varepsilon \langle 1$$

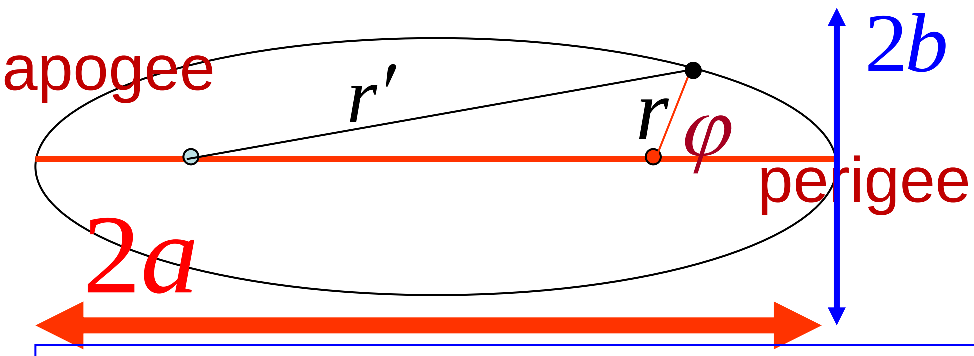


$$p = \frac{H^2}{\kappa} \ \& \ \varepsilon = \frac{A}{\kappa} = \sqrt{1+\frac{2EH^2}{\kappa^2}}$$

perigee $\quad r = r_{min}$ when $\varphi = 0$: $r_{min} = \dfrac{p}{1+\varepsilon}$

apogee $\quad r = r_{max}$ when $\varphi = \pi$: $r_{max} = \dfrac{p}{1-\varepsilon}$

$$r = \frac{p}{1+\varepsilon\cos\varphi}; \quad p = \frac{H^2}{\kappa}; \quad \varepsilon = \sqrt{1+\frac{2EH^2}{\kappa^2}}$$



At apogee/perigee:

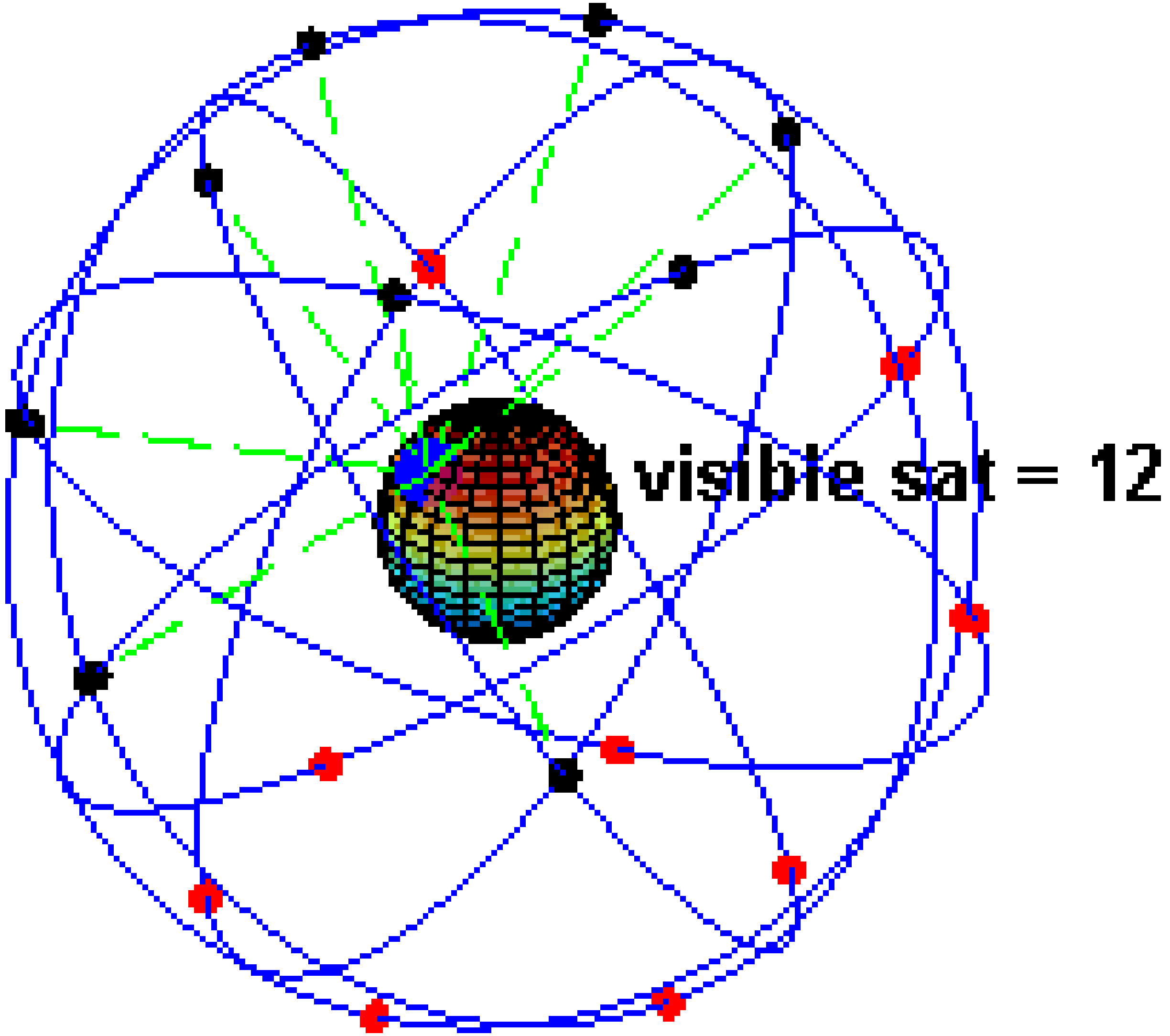
$r + r' = 2a$, constant

$$2a = r_{perigee} + r_{apogee} = \frac{p}{1-\varepsilon} + \frac{p}{1+\varepsilon} = \frac{2p}{1-\varepsilon^2}$$

$$p = a\left(1-\varepsilon^2\right)$$

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\varphi}$$

$$E = \frac{\kappa^2\left(\varepsilon^2-1\right)}{2H^2} = \frac{\kappa\left(\varepsilon^2-1\right)}{2p} \xrightarrow[\text{orbit}]{\text{for circular}} \frac{-\kappa}{2a}$$



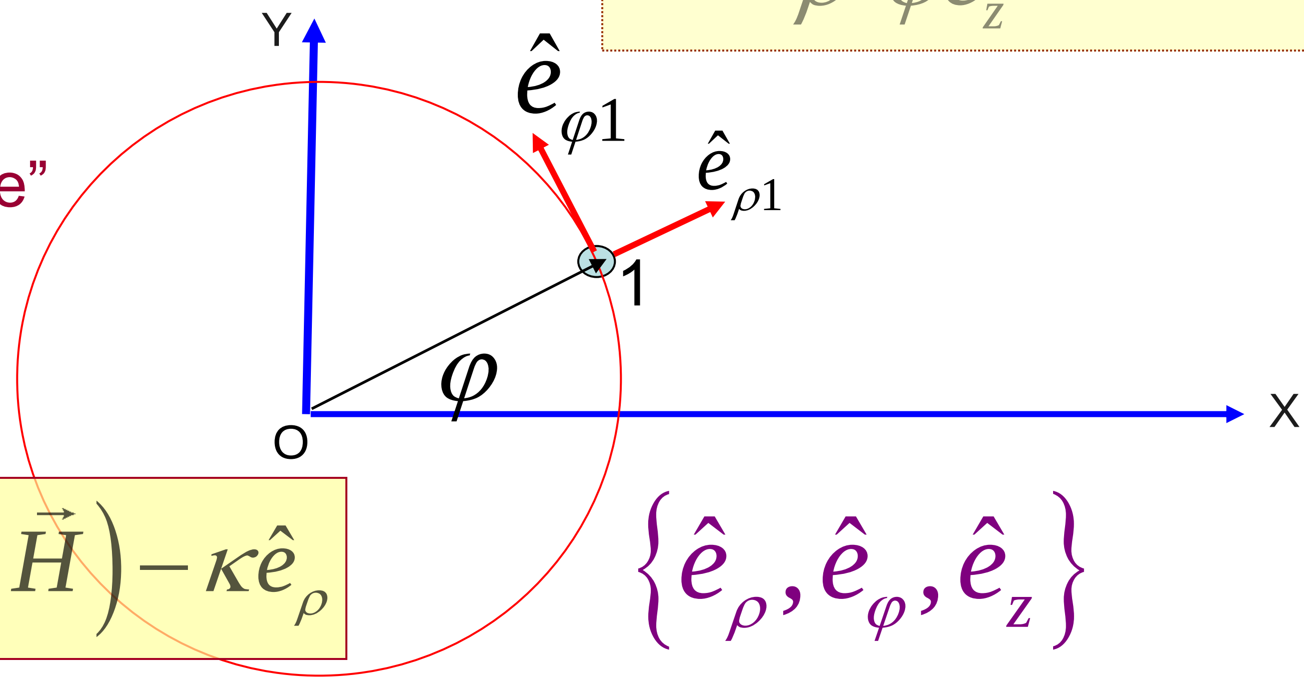
24 GPS satellites ---- Wikimedia Commons
http://en.wikipedia.org/wiki/File:ConstellationGPS.gif

$$\vec{\rho} = \rho \hat{e}_{\rho}$$

$$\vec{\mathrm{v}} = \dot{\rho}\hat{e}_{\rho} + \rho\dot{\varphi}\hat{e}_{\varphi}$$

$$\begin{aligned}\vec{H} &= \vec{r} \times \vec{\mathrm{v}} \\ &= \rho\hat{e}_{\rho} \times \left(\dot{\rho}\hat{e}_{\rho} + \rho\dot{\varphi}\hat{e}_{\varphi}\right) \\ &= \rho^{2}\dot{\varphi}\hat{e}_{z}\end{aligned}$$

"Unit Circle"

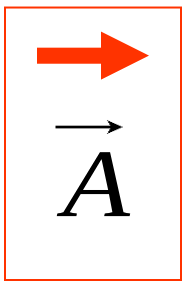


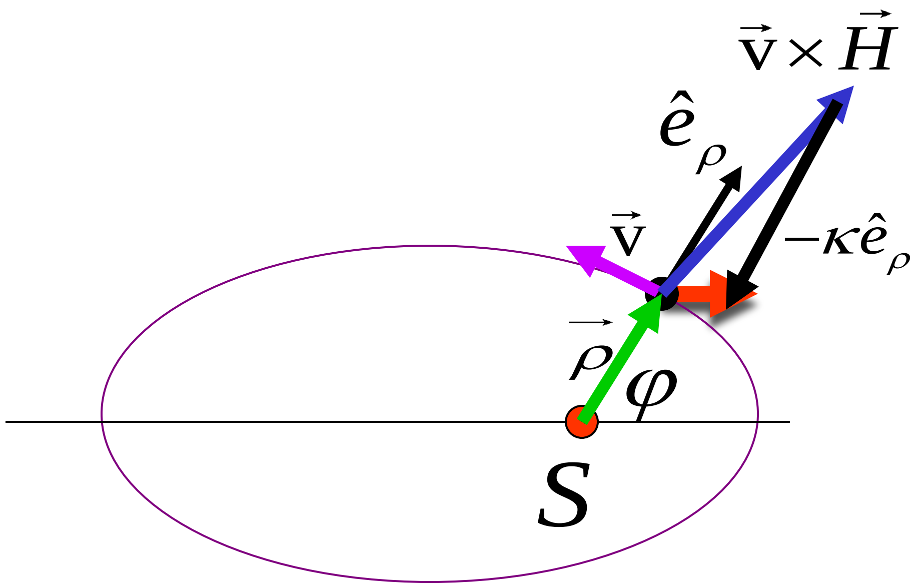
$$\vec{A} = \left(\vec{\mathrm{v}} \times \vec{H}\right) - \kappa\hat{e}_{\rho}$$

$$\left\{\hat{e}_{\rho}, \hat{e}_{\varphi}, \hat{e}_{z}\right\}$$

Plane/Cylindrical Polar Coordinate System

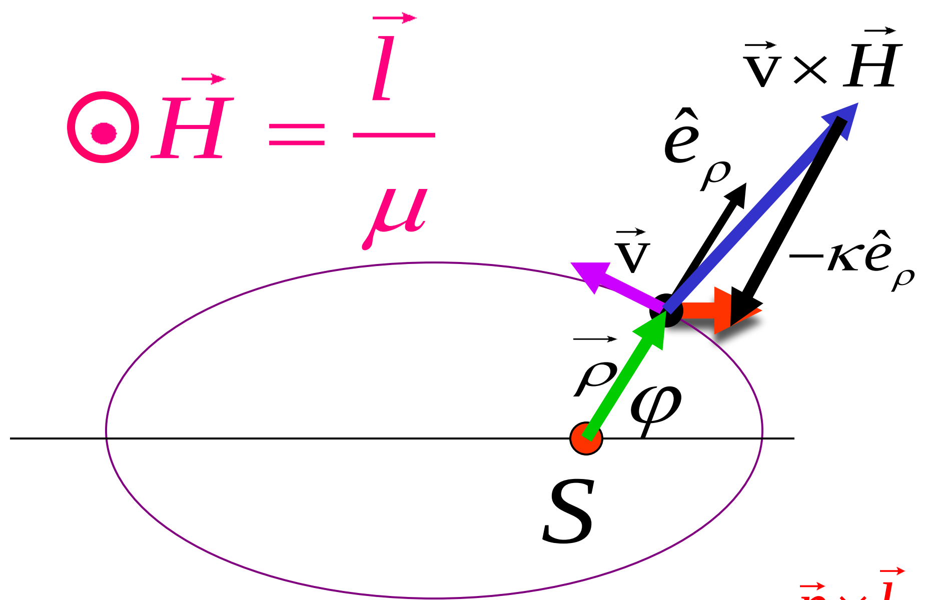
Laplace Runge Lenz Vector is constant for a strict ${}^{1}\!/_{r}$ potential.





$$\vec{A}=\left(\vec{v}\times\vec{H}\right)-\kappa\hat{e}_\rho$$

The $\odot\vec{H}$ (specific) angular momentum vector is *out* of the plane of this figure, toward us.



The Laplace-Runge-Lenz vector: $\vec{A} = \frac{\vec{p} \times \vec{l}}{\mu} - \mu\kappa\hat{e}_\rho$,

or alternatively defined in terms of the specific angular momentum $\vec{H}$ *as*

$\vec{A} = \vec{v} \times \vec{H} - \kappa\hat{e}_\rho$

$$\vec{A} = \left(\vec{\mathrm{v}} \times \vec{H}\right) - \kappa \hat{e}_\rho$$

$$\frac{d\vec{A}}{dt} = \left(\frac{d\vec{\mathrm{v}}}{dt} \times \vec{H} + \vec{\mathrm{v}} \times \frac{d\vec{H}}{dt}\right) - \kappa \frac{d\hat{e}_\rho}{dt}$$

Central Field Symmetry
Angular Momentum
is Conserved

$$\frac{d\vec{A}}{dt} = \left(\frac{d\vec{\mathrm{v}}}{dt} \times \vec{H}\right) - \kappa \frac{d\hat{e}_\rho}{dt}$$

$$But \quad \frac{d\hat{e}_\rho}{dt} = \frac{\partial \hat{e}_\rho}{\partial \varphi}\dot{\varphi} = \hat{e}_\varphi \dot{\varphi}$$

$$\frac{d\vec{A}}{dt} = \left(\frac{d\vec{\mathrm{v}}}{dt} \times \vec{H}\right) - \kappa \dot{\varphi} \hat{e}_\varphi$$

$$\vec{A} = \left(\vec{\mathrm{v}} \times \vec{H}\right) - \kappa \hat{e}_{\rho}$$

$$\frac{d\vec{A}}{dt} = \left(\frac{d\vec{\mathrm{v}}}{dt} \times \vec{H}\right) - \kappa \dot{\varphi} \hat{e}_{\varphi}$$

What is the form of the force per unit mass: $\frac{d\vec{\mathrm{v}}}{dt}$?

We must know the form of the interaction!

$$\vec{F} = m\vec{a} = m\frac{d\vec{\mathrm{v}}}{dt}$$

$$\boxed{\frac{v^2}{2}-\frac{\kappa}{r}=E}$$

The force per unit mass: $\dfrac{d\vec{v}}{dt}=-\dfrac{\kappa}{\rho^2}\hat{e}_\rho$

$$\frac{d\vec{A}}{dt}=\left(\frac{d\vec{v}}{dt}\times\vec{H}\right)-\kappa\dot{\varphi}\hat{e}_\varphi$$

$$\frac{d\vec{A}}{dt}=\left(-\frac{\kappa}{\rho^2}\hat{e}_\rho\times\left(\rho^2\dot{\varphi}\hat{e}_z\right)\right)-\kappa\dot{\varphi}\hat{e}_\varphi$$

$$\left\{\hat{e}_\rho,\hat{e}_\varphi,\hat{e}_z\right\}: right\ handed\ basis\ set$$

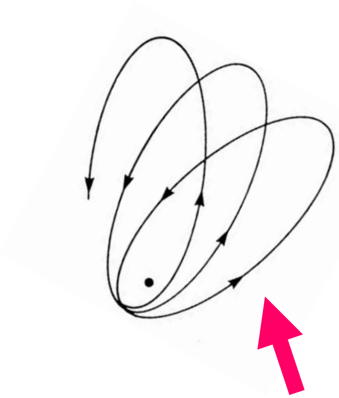
$$-\hat{e}_\rho\times\hat{e}_z=\hat{e}_\varphi$$

$$\frac{d\vec{A}}{dt}=\vec{0}$$

Two-body central field Kepler-Bohr problem,

Attractive force: inverse-square-law.

Rosette motion



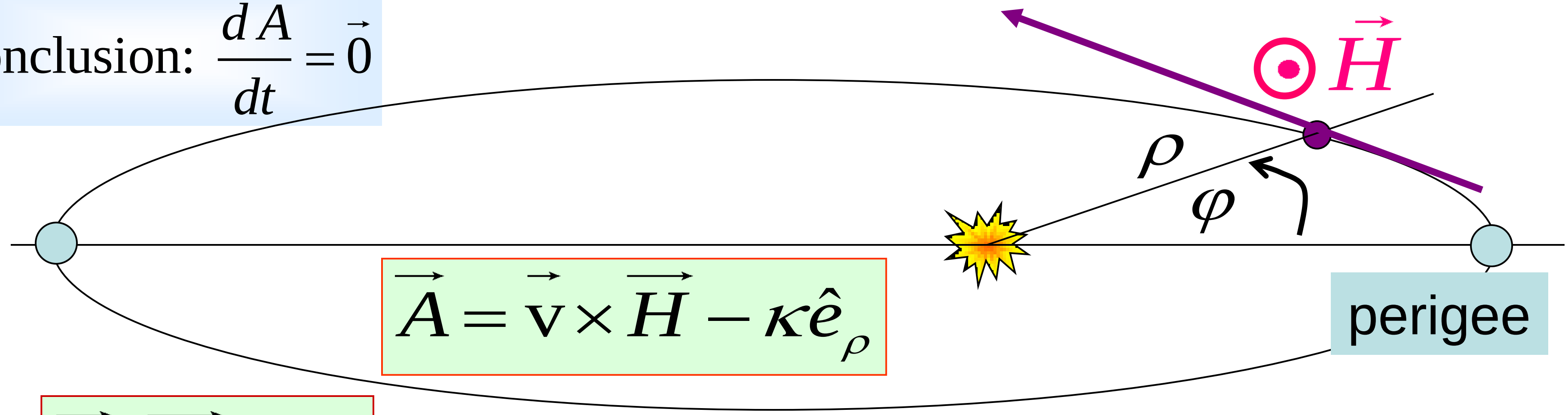
Angular momentum is conserved,
but major-axis not fixed.

For this potential and for the associated field:

no precession of orbit.

The constancy of the orbit suggests a conserved quantity and one must look for an associated symmetry.

Conclusion: $\frac{d\vec{A}}{dt} = \vec{0}$

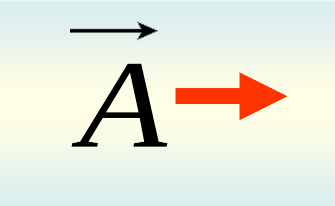


$$\vec{A} = \vec{v} \times \vec{H} - \kappa \hat{e}_\rho$$

$$\vec{A} \bullet \vec{H} = 0$$

LRL vector is in the plane of the orbit

Find the direction of $\vec{A}$ at the perigee.

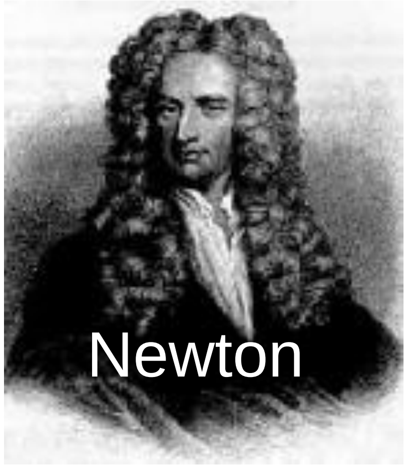


Direction of the LRL vector is: *focus to perigee.*

$\vec{A}$: Must remain constant

– *no matter where the planet is!*

This is precisely what FIXES the orbit!



Force (per unit mass): $\frac{d\vec{v}}{dt} = -\frac{\kappa}{\rho^2}\hat{e}_\rho$

Newton told us (but first, to Halley)!

Given the constancy related to the conservation of the LRL vector, could you have discovered the law of gravity?

If you did that, wouldn't you have discovered a law of nature?

Symmetry & Conservation Principles!

"It is now natural for us to try to derive the laws of nature and to test their validity by means of the laws of invariance, rather than to derive the laws of invariance from what we believe to be the laws of nature." - Eugene Wigner

Emmily Noether (1882 – 1935)

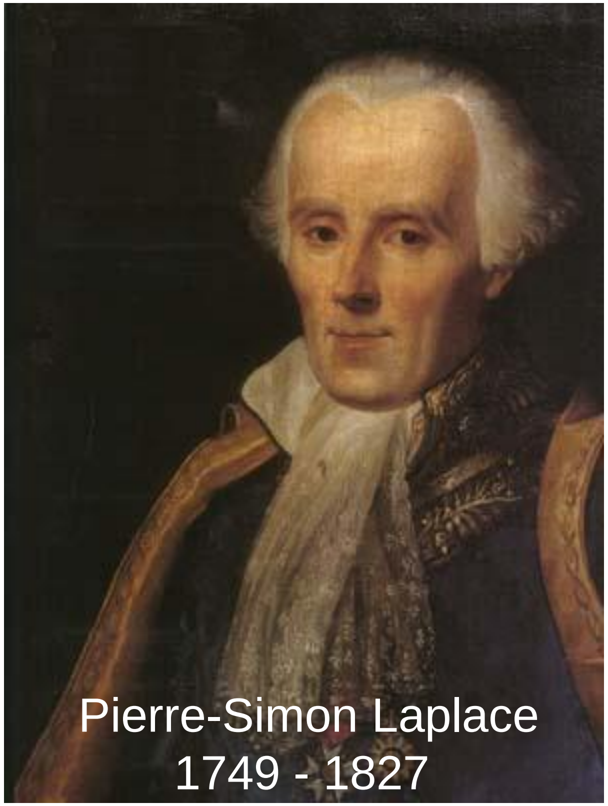
Eugene Paul Wigner (1902-1995)

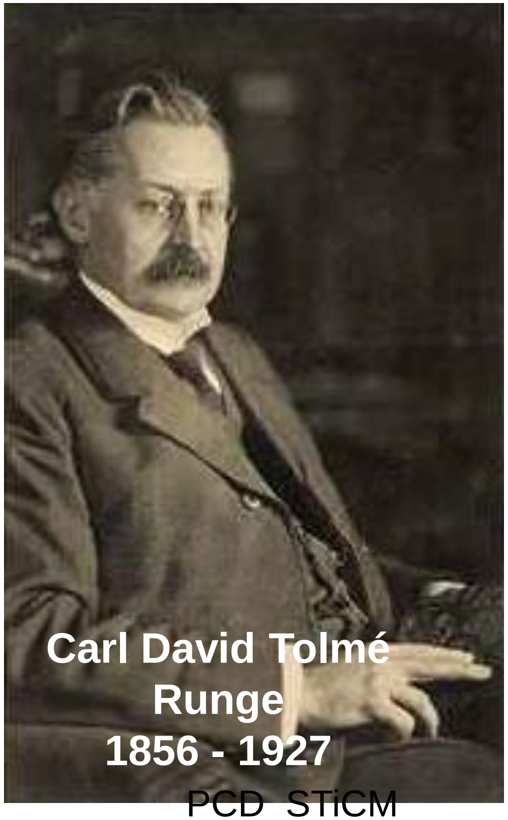
$$\vec{A} = \vec{v} \times \vec{H} - \kappa \hat{e}_\rho$$

Laplace Runge Lenz Vector :

constant for a strict $\frac{1}{r}$ potential.

Conservation law associated with '*dynamical / accidental*' symmetry.





Symmetry of the H atom: 'old' quantum theory. $E_n \sim n^{-2}$

PCD_STiCM

Further reading:

- Symmetry & Conservation laws play an important role in understanding the very frontiers of Physics.
- The implications go as far as testing the ‘standard’ model of physics, and exploring if there is any physics beyond the standards model.

Do visit:

Feynman's Messenger Lectures Online

AKA Project Tuva

http://www.fotuva.org/news/project_tuva.html

Useful references on ‘Symmetry & Conservation Laws’

P. C. Deshmukh and Shyamala Venkataraman
Obtaining Conservation Principles from Laws of Nature -- and the other way around!

Bulletin of the Indian Association of Physics Teachers, Vol. 3, 143-148 (2011)

P. C. Deshmukh and J. Libby
(a) Symmetry Principles and Conservation Laws in Atomic and Subatomic Physics -1
Resonance, 15, 832 (2010)
b) Symmetry Principles and Conservation Laws in Atomic and Subatomic Physics -2
Resonance, 15, 926 (2010)

Continuous Symmetries - Translation, Rotation

Dynamical Symmetries - LRL, Fock Symmetry
SO(4)

Discrete Symmetries
- P : Parity
- C : Charge Conjugation
- T : Time Reversal

Lorentz symmetry: associated with the PCT symmetry.

PCT theorem (Wolfgang Pauli)

No experiment has revealed any violation of PCT symmetry.

This is predicted by the Standard Model of particle physics.

The Standard Model today

Elementary particles

		First family	Second family	Third family
Leptons		electron neutrino	muon neutrino	tau neutrino
		electron	muon	tau
Quarks		up	charm	top
		down	strange	bottom

Forces	Messenger particles
electromagnetic force	photon
weak force	W, Z
strong force	gluons

Higgs?

The 'standard model' unifies all the fundamental building blocks of matter, and three of the four fundamental forces.

To complete the Model a new particle is needed – the Higgs Boson – that the physics community hopes to find in the new built accelerator LHC at CERN in Geneva.

2008 Nobel Prize in Physics

"for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics"

Yoichiro Nambu
½ prize
Enrico Fermi Institute, Chicago, USA

"for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature"

Makoto Kobayashi
¼ prize
High Energy Accelerator Research Organization, Tsukuba, Japan

Toshihide Maskawa
¼ prize
Kyoto Sangyo Univ, KyotoJapan

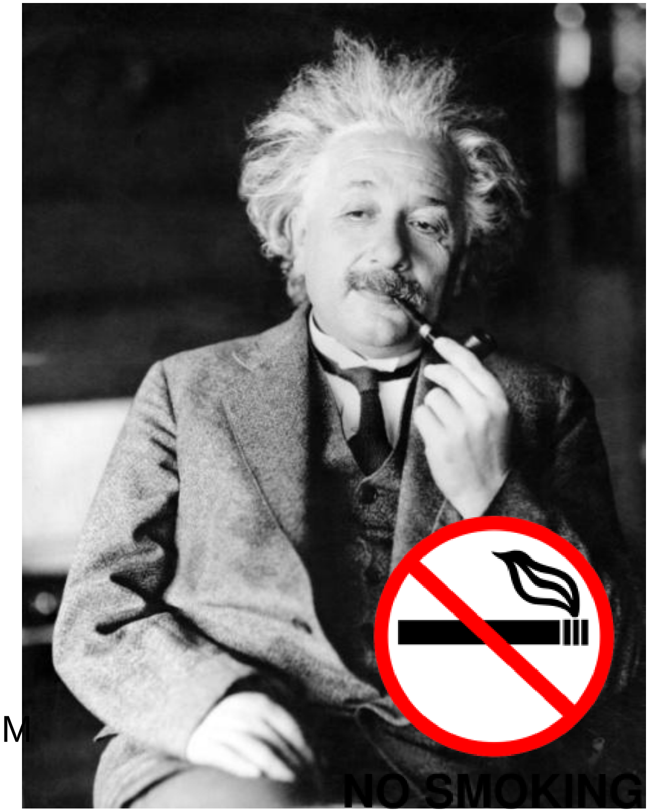
"every symmetry in nature yields

a conservation law

and conversely,

every conservation law reveals

an underlying symmetry'.



Noether's theorem

"In the judgment of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began."

"In the realm of algebra, … , she discovered methods which have proved of enormous importance …. "

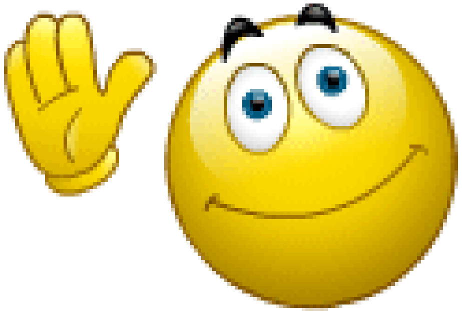
"………. Her unselfish, significant work over a period of many years was rewarded by the new rulers of Germany with a dismissal, which cost her the means of maintaining her simple life and the opportunity to carry on her mathematical studies……"

ALBERT EINSTEIN. Princeton University, May 1, 1935.
New York Times May 5, 1935. *Excerpts*

http://www-history.mcs.st-and.ac.uk/history/Obits2/Noether_Emmy_Einstein.html

We will take a Break…

…… Any questions ?



pcd@physics.iitm.ac.in

Next: Unit 5: Inertial and non-inertial reference frames.

Moving coordinate systems. Pseudo forces.
Inertial and non-inertial reference frames.

'Deterministic' cause-effect relations in inertial frame, and their *modifications* in a *non-inertial* frame.